Black holes quasi-normal modes

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Greetings

I wish to thank the professor Valeria Ferrari to have give me a subject to study in the frame of the internships, and for her advices. I would like to thank her too for the quality of the lessons of General Relativity and Gravitational Waves, stars and black holes.

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Abstract

The gravitational waves, produced by the massive objects with strong asymmetry, are the subject of important research. Indeed, they are predicted by the theory of the relativity of Albert Einstein, like a consequence of the equivalence principle. These waves are totally characterized by their source, and are entirely linked to its mass and shape. Their recent detection is a good point for the validity of the relativistic theory of Einstein, and give us some responses in cosmology or astrophysics.
The topic of the following thesis is the characterization of the gravitational waves quasi-normal modes.

The study is firstly based on mathematics side with the expansion of the Einstein equations on the tensorial basis of the spherical harmonics, the problem being spherically symmetric. Then, we study the quasi-normal modes, and finally, we make the same expansion with a source term.

Résumé

Les ondes gravitationnelles, produites par des objets massifs à forte assymétrie, font l'objet de vives recherches. En effet, elles sont prédites par la théorie de la relativité d'Albert Einstein, comme conséquence du principe d'équivalence. Ces ondes sont entièrement caractérisées par leur source, puisqu'elles dépendent de sa masse ainsi que de sa forme. Leur récente détection est un renfort à la validité de la théorie relativiste d'Einstein, et nous apporte aussi plusieurs réponses dans des domaines variés comme la cosmologie ou l'astrophysique.
Le sujet de la thèse suivante est la caractérisation des modes quasi-normaux des ondes gravitationnelles.

L'étude est d'abord basée sur un aspect très mathématique avec le développement des équations de Einstein perturbées sur la base tensorielle des harmoniques sphériques, le problème étant à symétrie sphérique. Nous mettrons ensuite en exergue les modes quasi-normaux, et enfin nous ferons le même développement avec un terme source.

Sommario

Le onde gravitazionali, prodotte per degli oggetti massivi con forte asimmetria, fanno l'oggetto di importante ricerche. Sono previste per la teoria della relatività di Albert Einstein, come consequenza del principio di equivalenza. Queste onde sono interamente caratterizzate per la loro sorgente, in fatti, sono legate alla sua massa e la sua forma. La loro recente rivelazione è un conforto alla validità della teoria relativistica di Einstein, e ci da anche plurale risposte nella cosmologia o l'astrofisica.
Il soggetto della tesi seguente è la caraterizzazione dei modi quasi-normali delle onde gravitazionali.

Lo studio è primo basato su un aspetto molto matematico con il sviluppo degli equazioni di Eintein perturbate sulla basa tensoriale degli armoniche sferiche, il problemo essendo a simmetria sferica. Dopo, studiamo i modi quasi-normali, e alle fine, facciamo lo stesso sviluppo con un termine sorgente.

Contents

1 Introduction

The recent detection of the gravitational waves by the interferometer LIGO has permit to verify the gravitational theory of the general relativity. Indeed, these ones were predicted like a consequence of this theory. A such detection, has several results. The object detected corresponds to a normal mode of a massive object, a black hole. The gravitational waves were predicted by the Einstein theory, and are observables in the case of massive and asymmetric object ; in fact the intensity of waves is directly linked to the mass and the quadripolar moment of its source. How can we characterize these waves ?

In the astrophysical field, there are two main consequences, the first one consists on the understanding of black holes ; indeed, produced by the massive objects, the gravitational waves are totally linked to their source. The waves detected were coming from a black hole of 30-40 solar masses whereas the mass was expected like ten solar masses, which implies the existence of supermassive stars never observed, why ? Beyond the black holes, these waves teach us about the galactic centers formed by a supermassive black hole ; for example, in the case of the Milky Way, this one weighs more than four million solar masses.
Concerning the cosmology, the waves bring some responses to the question of the Big Bang, the inflation and the expansion of the Universe. Indeed, the deformation of the space-time induced by the expansion of the Universe produces some gravitational waves which can be detected. Thus, we can learn some things about the Big Bang, detecting some waves coming from this era. Since the waves come from massive object, in the hypothesis of the existence of the dark matter, they could highlight the dark matter clusters in the Universe, and thus permit to detect it.
Based on the general relativity, like Einstein equations and the basis concepts of the special relativity like the coordinate time, the following thesis will study the production and the modes of the gravitational waves.

In a first part, since the background of the problem is spherically symmetric, we will expand all the quantities on the tensorial spherical harmonics basis. Then, the second purpose will concern the calculation of the perturbed metric caused by the introduction of a particle falling radially on the black hole, which corresponds to the source term of the gravitational waves ; because of the asymmetry.

2 Definitions of the problem

2.1 Elements of general relativity

2.1.1 Metric of the Schwarzschild black hole

To describe the black hole oscillations at the origin of the gravitational waves, we have to define the metric of the perturbed black hole :

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \ ;$$

with $g^0_{\mu\nu}$ is the exact solution of Einstein equations. In the case of a Schwarzschild black hole, it is the Schwarzschild solution : the static metric, asymptotically the Minkowski metric $\eta_{\mu\nu}$.

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This solution is the only spherically and asymptotically flat solution of the vacuum Einstein field equations for a non-rotating object.
The line-element of the Schwarzschild metric is expressed by :

$$\mathrm{ds}^2 = \text{-}(1 \text{ - } \tfrac{r_s}{r})\ \mathrm{c}^2\ \mathrm{dt}^2 + \tfrac{1}{1-\frac{r_S}{r}}\ \mathrm{dr}^2 + \mathrm{r}^2\ (\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2),$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, G the gravitational constant, M the mass of the object and c the light velocity in vacuum. In the following parts, we will use the natural units, i.e. G=c=1.
The Schwarzschild radius defines the event horizon of a Schwarzschild black hole. For a distance from the black hole less than this radius, we can't have information about the events inside this volume. The spherical surface of radius r_s is a spacelike surface, i.e. it can be through only in one direction, and in our case for r decreasing.

Einstein's equations, which link the behaviour of the matter and the stress-energy tensor, are :

$$\mathrm{G}_{\mu\nu} = \tfrac{8\pi G}{C^4}\ \mathrm{T}_{\mu\nu}$$

with $\mathrm{G}_{\mu\nu} = \mathrm{R}_{\mu\nu}$ - $\frac{1}{2}$ $\mathrm{g}_{\mu\nu}$ R, where $\mathrm{R}_{\mu\nu}$ is the Ricci's tensor and R the curvature of the space.
$\mathrm{T}_{\mu\nu}$ is the stress-energy tensor. For a system of n particles, it is defined by

$$\mathrm{T}^{\mu\nu} = \mathrm{c}\ \Sigma_n \int \tfrac{1}{\sqrt{-g}} p^\mu_n \tfrac{dx^\beta_n}{d\tau_n} \delta^4(x - x_n)\, \mathrm{d}\tau_n$$

with g the determinant of the metric, and p_n and τ_n, respectively the momentum and the proper time of the particle n.
In the case of vacuum, $\mathrm{T}^{\mu\nu}$ is null, and the Einstein's equations are just :

$$\mathrm{G}_{\mu\nu} = 0$$

Now, we have the metric used to describe a Schwarszchild's black hole. There exists two important things for this metric :

-the first one is the asymptotical limit $\eta_{\mu\nu}$, the minkowskian space of dimension 4
-the second one is about the two singularities, $\mathrm{r} = \mathrm{r}_s$ which defines the event horizon, and $\mathrm{r} \longrightarrow 0$ which is a mathematical singularity.

2.1.2 Einstein's equations for black hole oscillations

In the previous section, we have seen that the metric for the oscillations is $\mathrm{g}_{\mu\nu} = \mathrm{g}^0_{\mu\nu} + \mathrm{h}_{\mu\nu}$, with $\mathrm{g}^0_{\mu\nu}$ the Schwarzschild's metric. In our theory, we will consider $\mathrm{h}_{\mu\nu}$ like a perturbation of the Schwarzschild metric, thus we have $|h_{\mu\nu}| << |g^0_{\mu\nu}|$.
The idea is to solve the Einstein equations for the perturbation. The Einstein tensor for the perturbed metric is at the first order

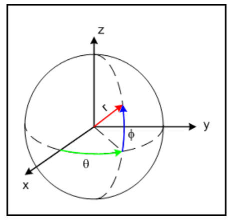

Figure 1: Spherical coordinates.

$$G_{\mu\nu} = G_{\mu\nu}(g^0_{\mu\nu}) + \delta\ G_{\mu\nu}(h) + O(h^2)$$

According to the previous section, in vacuum, $G_{\mu\nu}(g^0_{\mu\nu}) = 0$, and thus the Einstein's equations for the perturbed metric tensor are :

$$\delta\ G_{\mu\nu}(h) = 0, \text{ which are linear in } h_{\mu\nu}.$$

We can express the Einstein tensor as $G_{\mu\nu}= R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$ (see annex 6.1). We recall the expression of the Riemann tensor :

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\ [g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}]$$

where $_{,\alpha}$ is the ordinary derivative with respect to the coordinate α.
The Ricci tensor is defined by $R_{\beta\delta} = R^{\alpha}_{\beta\alpha\delta} = g^{\alpha\mu}\ R_{\mu\beta\alpha\delta}$.
The curvature of the metric is defined by $R = g^{\alpha\beta}\ R_{\alpha\beta}$.

At the first order in h, we expand the expression of R_{ij} and $g_{ij}R$, using $g_{ij} = g^0_{ij} + h_{ij}$ and $g^{ij} = g^{0ij} - h^{ij}$.

After some calculations (see annex 6.2), we obtain the explicit expression :
$\delta G_{\mu\nu}(h) = -\frac{1}{2}[h^{;\alpha}_{\mu\nu;\alpha} - (h^{;\alpha}_{\mu\alpha})_{;\nu} - (h^{;\alpha}_{\nu\alpha})_{;\mu} + 2\ R_{\alpha\mu\beta\nu}(g^0)h^{\alpha\beta} + h^{\alpha}_{\alpha;\mu\nu} - R^{\alpha}_{\nu}(g^0)h_{\mu\alpha} - R^{\alpha}_{\mu}(g^0)h_{\nu\alpha} + g^0_{\mu\nu}((h^{;\alpha}_{\mu\alpha})^{;\mu} - (h^{\alpha}_{\alpha;\beta})^{;\beta} - R^{\alpha\beta}(g^0)h_{\alpha\beta} + R(g^0)h_{\mu\nu}] = 0$.

2.2 Expansion in spherical harmonics

The background of the problem is spherically symmetric since the system is a massive sphere. Indeed, each quantity can be expressed as a function of the coordinate time **t**, the radial position **r**, the equatorial angle $\boldsymbol{\theta}$, and the azimuthal angle $\boldsymbol{\phi}$. For **t** fixed, the geometrical coordinates (**r**, $\boldsymbol{\theta}$, $\boldsymbol{\phi}$) are defined on the following figure from **x**, **y** and **z**, which are the cartesian coordinates of the real space at three dimensions.

Hence, we need to expand a tensor of rank 2 in spherical harmonics (see annex 6.3). This is possible according to Regge and Wheeler (1957) and Zerilli (1970).
This is possible for quadrivector (see annex 6.4), and also for tensor of rank 2.

The general idea is to use the procedure used for quadrivectors for the tensors. We have to find a basis for tensors.
According to Zerelli (1970), the basis is formed by ten tensors harmonics, given by the following expressions.

$$\begin{cases} a_{lm} = [\boldsymbol{e_r}\ \boldsymbol{e_r}\ \mathrm{Y}_{lm}] \\ b_{lm} = \sqrt{2}\ \mathrm{n(l)\ r}\ [\boldsymbol{e_r}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}] \\ c_{lm} = \sqrt{2}\ \mathrm{n(l)\ r}\ [\boldsymbol{e_r}\ \mathrm{L\ Y}_{lm}] \\ d_{lm} = \sqrt{2}\ \mathrm{m(l)\ r}\ ([\mathrm{L}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}] + \frac{1}{r}\ [\boldsymbol{e_r}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}] \\ f_{lm} = \frac{1}{\sqrt{2}}\ \mathrm{m(l)}\ (\boldsymbol{e_{lm}} + \boldsymbol{h_{lm}}) \\ g_{lm} = -\frac{1}{\sqrt{2}}\ \mathrm{n^2(l)}\ (\boldsymbol{e_{lm}}\ \text{-}\ \boldsymbol{h_{lm}}) \\ a^{(0)}_{lm} = [\boldsymbol{e_t}\ \boldsymbol{e_t}\ \mathrm{Y}_{lm}] \\ a^{(1)}_{lm} = \sqrt{2}\ [\boldsymbol{e_t}\ \boldsymbol{e_r}\ \mathrm{Y}_{lm}] \\ b^{(0)}_{lm} = \sqrt{2}\ \mathrm{n(l)\ r}\ [\boldsymbol{e_t}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}] \\ c^{(0)}_{lm} = \sqrt{2}\ \mathrm{n(l)}\ [\boldsymbol{e_t}\ \mathrm{L\ Y}_{lm}] \end{cases} \tag{1}$$

Where
$\boldsymbol{e_{lm}} = \mathrm{r}^2\ ([\boldsymbol{\nabla}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}] + \frac{2}{r}\ [\boldsymbol{e_r}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}])$
$\boldsymbol{h_{lm}} = = [\mathrm{L\ L\ Y}_{lm}] + \mathrm{r}\ [\boldsymbol{e_r}\ \boldsymbol{\nabla}\ \mathrm{Y}_{lm}]$
$\mathrm{n(l)} = \frac{1}{\sqrt{l(l+1)}}$
$\mathrm{m(l)} = \frac{1}{\sqrt{l(l+1)(l-1)(l+2)}}$
Thus, we can extend any symmetric tensor T(t, r, θ, ϕ) as follows :

$$\Sigma_{lm}\ [\ \mathrm{A}^{(0)}_{lm}\ \mathrm{a}^{(0)}_{lm} + {}^{(1)}_{lm}\ \mathrm{a}^{(1)}_{lm} + \mathrm{A}_{lm}\ \mathrm{a}_{lm} + \mathrm{B}^{(0)}_{lm}\ \mathrm{b}^{(0)}_{lm} + \mathrm{B}_{lm}\ \mathrm{b}_{lm} + \mathrm{Q}^{(0]}_{lm}\ \mathrm{c}^{(0)}_{lm} + \mathrm{Q}_{lm}\ \mathrm{c}_{lm} + \mathrm{G}_{lm}\ \mathrm{g}_{lm} + \mathrm{D}_{lm}\ \mathrm{d}_{lm} + \mathrm{F}_{lm}\ \mathrm{f}_{lm}\ ]$$

And the coefficients are found by the relations :

$$\mathrm{A}^{(0)}_{lm}\ (\mathrm{t, r}) = \int a^{(0)\mu\nu *}_{lm} T_{\mu\nu} sin\theta\, \mathrm{d}\theta \mathrm{d}\phi$$

An important result is the following : the coefficients depend only of t and r, not of the angles. The angular dependence is in the spherical harmonic basis.

2.3 Parity

A parity transformation is a simultaneous inversion of the cartesian axis. In spherical coordinates, the transformation is the following :

$$(\theta, \phi) = (\pi - \theta, \pi + \theta)$$

For these transformations, there exists a relation for spherical harmonics :

$$\mathrm{Y}_{lm}(\pi - \theta, \pi + \theta) = (\text{-}1)^l \mathrm{Y}_{lm}(\theta, \phi)$$

since $\mathrm{P}_{lm}(\text{-x}) = (\text{-}1)^{l+m}\mathrm{P}_{lm}(\mathrm{x})$.

The factor $(\text{-}1)^l$ is the parity of the spherical harmonic : if l is even, the parity is said **polar**, if l is odd, the parity is said **axial**.
The spherical harmonic tensors a_{lm}, b_{lm}, f_{lm}, g_{lm}, $\mathrm{a}^{(0)}_{lm}$, $\mathrm{a}^{(1)}_{lm}$ and $\mathrm{b}^{(0)}_{lm}$ follow a law with l even, so these tensors are polar.
The tensors c_{lm}, d_{lm} and $\mathrm{c}^{(0)}_{lm}$ have a transformation with l odd, and thus these tensors are axials.
Like we have developed a tensor on the spherical harmonic basis, we can decompose a tensor $\mathrm{T} = \mathrm{T}^{ax} + \mathrm{T}^{pol}$.
Now we have the basis to solve our problem.

3 Black hole oscillations

3.1 Expansion in spherical harmonics

The background of the problem is spherically symmetric, by the previous section, we can expand a tensor in spherical harmonics, and decompose it in an axial part and a polar part. This is almost true for the perturbation $h_{\mu\nu}$, and we have :

$$h_{\mu\nu} = h^{ax}_{\mu\nu} + h^{pol}_{\mu\nu}$$

with,
$h^{ax} = \Sigma_{lm}\ [Q^{(0]}_{lm}\ c^{(0)}_{lm} + Q_{lm}\ c_{lm} + D_{lm}\ d_{lm}]$
$h^{pol} = \Sigma_{lm}\ [\ A^{(0)}_{lm}\ a^{(0)}_{lm} + {}^{(1)}_{lm}\ a^{(1)}_{lm} + A_{lm}\ a_{lm} + B^{(0)}_{lm}\ b^{(0)}_{lm} + B_{lm}\ b_{lm} + G_{lm}\ g_{lm} + F_{lm}\ f_{lm}\ ]$

The coefficients of the expansion have to be found solving Einstein's equations. We can write :

$$\begin{cases} D_{lm}(t, r) = -\frac{i}{2r^2}\sqrt{2l(l+1)(l-1)(l+2)}\ h_{2lm}(t, r) \\ Q^{(0)}_{lm}(t, r) = -\frac{i}{r}\sqrt{2l(l+1)}\ h^{ax}_{0lm}(t, r) \\ Q_{lm}(t, r) = \frac{i}{r}\sqrt{2l(l+1)}\ h^{ax}_{1lm}(t, r) \end{cases} \tag{2}$$

We have three axial functions : $h^{ax}_0(t, r)$, $h^{ax}_1(t, r)$ and $h_2(t, r)$.

$$\begin{cases} F_{lm}(t, r) = 2\ \frac{\sqrt{l(l+1)(l-1)(l+2)}}{2}\ V_{lm}(t, r) \\ A^{(0)}_{lm}(t, r) = 2e^{2\nu}\ N_{lm}(t, r) \\ A^{(1)}_{lm}(t, r) = \sqrt{2}\ H_{1lm}(t, r) \\ A_{lm}(t, r) = -2e^{2\lambda}\ L_{lm}(t, r) \\ B^{(0)}_{lm}(t, r) = \frac{\sqrt{2l(l+1)}}{r}\ h_{0lm}(t, r) \\ B_{lm}(t, r) = \frac{\sqrt{2l(l+1)}}{r}\ h_{1lm}(t, r) \\ G_{lm}(t, r) = -2\sqrt{2}\ T_{lm}(t, r) + \frac{2l(l+1)}{\sqrt{2}}\ V_{lm}(t, r) \end{cases} \tag{3}$$

We have seven polar functions : $V(t, r)$, $N(t, r)$, $H_1(t, r)$, $L(t, r)$, $h_0(t, r)$, $h_1(t, r)$ and $T(t, r)$. The functions $e^{2\nu}$ and $e^{2\lambda}$ are the coefficients of the unperturbed metric of Schwarzschild g^0_{00} and g^0_{rr}.

$$e^{2\nu} = (1 - \frac{r_s}{r}),\ e^{2\lambda} = \frac{1}{1-\frac{r_s}{r}},\ r_s \text{ is the Schwarzschild's radius.}$$

Having computed the expression of $h_{\mu\nu}$ on the spherical harmonic basis, we obtain :

$$h^{ax} = \Sigma_{lm}\begin{pmatrix} 0 & h^{ax}_0\ \sin\theta\partial_\theta Y_{lm} & 0 & -h^{ax}_0\ \frac{1}{sin\theta}\partial_\phi Y_{lm} \\ h^{ax}_0\ \sin\theta\partial_\theta Y_{lm} & -h_2\ \frac{1}{2}\ \sin\theta\ X_{lm} & h^{ax}_1\ \sin\theta\partial_\theta & -h_2\ \frac{1}{2}\ \sin\theta\ W_{lm} \\ 0 & h^{ax}_1\ \sin\theta\partial_\theta & 0 & -h^{ax}_1\ \frac{1}{sin\theta}\partial_\phi Y_{lm} \\ -h^{ax}_0\ \frac{1}{sin\theta}\partial_\phi Y_{lm} & -h_2\ \frac{1}{2}\ \sin\theta\ W_{lm} & -h^{ax}_1\ \frac{1}{sin\theta}\partial_\phi Y_{lm} & h_2\ \frac{1}{2}\ \sin\theta\ X_{lm} \end{pmatrix}$$

$$h^{pol} = \Sigma_{lm}\begin{pmatrix} 2e^{2\nu}\ N\ Y_{lm} & -h_0\ \partial_\phi Y_{lm} & -H_1\ Y_{lm} & -h_0\ \partial_\theta Y_{lm} \\ -h_0\ \partial_\phi Y_{lm} & -2r^2\ \sin^2\theta\ H_{11} & h_1\ \partial_\phi Y_{lm} & -r^2\ V\ X_{lm} \\ -H_1\ Y_{lm} & h_1\ \partial_\phi Y_{lm} & -2e^{2\lambda}\ L\ Y_{lm} & h_1\ \partial_\theta Y_{lm} \\ -h_0\ \partial_\theta Y_{lm} & -r^2\ V\ X_{lm} & h_1\ \partial_\theta Y_{lm} & -2r^2\ H_{33} \end{pmatrix}$$

With
$X_{lm} = 2\ \partial_\phi[\partial_\theta - \cot\theta]\ Y_{lm}$
$W_{lm} = [\partial^2_\theta - \cot\theta\ \partial_\theta - \frac{1}{sin^2\theta}\ \partial^2_\phi\ ]\ Y_{lm}$
$H_{11} = [T + V(\frac{1}{sin^2\theta}\ \partial^2_\phi + \cot\theta\ \partial_\theta)]\ Y_{lm}$
$H_{33} = [T + V\ \partial^2_\theta]\ Y_{lm}$

At this point, according to the general relativity, we have the freedom of making a coordinate transformation, i.e. to choose a gauge, in which to solve Einstein's equations. This is called a choice of gauge.

3.2 The choice of the gauge

This choice of gauge, or freedom on the coordinate allos to redjuce the number of unknown functions. To choose an appropriate gauge, we can make an infinitesimal coordinate transformation :

$$x'^j = x^j + \xi^j,$$

where ξ^j is an infinitesimal vector.
We need to compute the perturbed metric according to this transformation.
We know that the transformation of the metric is $g'_{\mu\nu} = \Lambda^i_\mu \Lambda^j_\nu g_{ij}$, where Λ is the matrix of the transformation defined by $\Lambda^i_j = \frac{dx^i}{dx'^j}$.
So,

$$g'_{ij} = \frac{dx^a}{dx'^i} \frac{dx^b}{dx'^j} g_{ab}$$

Since $dx^a = dx'^a - d\,\xi^a$ and $\frac{dx^i}{dx^j} = \delta^i_j$, the Kronecker's symbol, we have at the first order in ξ (infinitesimal transformation),

$$g'_{ij} = \delta^a_i \delta^b_j g_{ab} - \delta^b_j \xi^a_{,i} g_{ab} - \delta^a_i \xi^b_{,j} g_{ab}$$

Using the fact that for a vector V^a we have $V_b = V^a g_{ab}$,
The preceeding equation becomes

$$g'_{ij} = g_{ij} - \xi_{a,i} - \xi_{b,j}$$

Like the covariant derivative and the normal derivative coincide by the equivalence principle, we can use this result for the perturbed metric, noting by $_{;i}$ the covariant derivative according to the coordinate i, and this is true for every locally inertial frame,

$$h'_{ij} = h_{ij} - \xi_{i;j} - \xi_{j;i} = h_{ij} - \xi_{i,j} - \xi_{j,i} + 2\Gamma^k_{ij} \xi_k$$

This equality comes from the relation between covariant and normal derivative $\xi_{i;j} = \xi_{i,j} - \Gamma^k_{ij} \xi_k$, and the fact that Γ^k_{ij}, the Cristoffel symbol, is symmetric under the exchange between i and j.
Since we have expand a tensor in spherical harmonics, we have to expand also the vector $\boldsymbol{\xi}$ in vector spherical harmonics, and analogously, we have the following decomposition, between a polar part and an axial part :

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{pol} + \boldsymbol{\xi}^{ax}$$

where

$$\begin{cases} \xi^{pol} = (\, f^{(0)}_{lm} Y_{lm},\, -f^{(2)}_{lm} \partial_\phi Y_{lm},\, -f^{(3)}_{lm} Y_{lm},\, -f^{(2)}_{lm} \partial_\theta Y_{lm} \,) \\ \xi^{ax} = (0,\, i\, r\, f^{(1)}_{lm} \sin\theta\, \partial_\theta Y_{lm},\, 0,\, -i\, r\, f^{(1)}_{lm} \frac{1}{sin\theta} \partial_\phi Y_{lm} \,) \end{cases} \tag{4}$$

We can set four new functions :

$$\begin{cases} M_0\ (t,r) = f^{(0)}_{lm}\ (t, r) \longrightarrow \text{polar} \\ M_1\ (t,r) = -f^{(3)}_{lm}\ (t, r) \longrightarrow \text{polar} \\ M_2\ (t,r) = -f^{(2)}_{lm}\ (t, r) \longrightarrow \text{polar} \\ M_3\ (t,r) = -\, i\, r\, f^{(1)}_{lm}\ (t, r) \longrightarrow \text{axial} \end{cases} \tag{5}$$

To choose a gauge, we will study separately the two components of the metric, choosing a polar gauge and an axial gauge.

3.2.1 Polar part of the perturbed metric

We consider only the even perturbations. Using the same transformation than before, having computed the different Cristoffel's symbol,

$$\begin{cases} h'_{tt} = h_{tt} - 2\, M_{0,t} Y_{lm} + 2\, \nu_{,r}\, e^{2(\nu-\lambda)} M_1 Y_{lm} \\ h'_{t\phi} = h_{t\phi} - M_0\, \partial_\phi Y_{lm} - M_{2,t}\, \partial_\phi Y_{lm} \\ h'_{tr} = h_{tr} - M_{0,r} Y_{lm} - M_{1,t} Y_{lm} + 2\, \nu_{,r} M_0 Y_{lm} \\ h'_{t\theta} = h_{t\theta} - M_0\, \partial_\theta Y_{lm} - M_{2,t}\, \partial_\theta Y_{lm} \\ h'_{\phi\phi} = h_{\phi\phi} - 2\, M_2\, \partial^2_\phi Y_{lm} - 2[r\, e^{-2\lambda} M_1 \sin^2\theta\, Y_{lm} + M_2 \sin\theta \cos\theta\, \partial_\theta Y_{lm}\,] \\ h'_{r\phi} = h_{r\phi} - M_{2,r}\, \partial_\phi Y_{lm} - M_1\, \partial_\phi Y_{lm} + \frac{2}{r} M_2\, \partial_\phi Y_{lm} \\ h'_{\phi\theta} = h_{\phi\theta} - 2\, M_2\, \partial_\theta\partial_\phi Y_{lm} + 2\, M_2 \cot\theta\, \partial_\phi Y_{lm} \\ h'_{rr} = h_{rr} - 2\, M_{1,r} Y_{lm} + 2\, \lambda_{,r} M_1 Y_{lm} \\ h'_{r\theta} = h_{r\theta} - M_{2,r}\, \partial_\theta Y_{lm} + \frac{2}{r} M_2\, \partial_\theta Y_{lm} \end{cases} \tag{6}$$

Consequently to the gauge transformation, all polar coefficients are transformed as follows :

$$\begin{cases} N' = N - e^{-2\nu}\ \mathrm{M}_{0,t} + \nu_{,r}\ \mathrm{e}^{-2\lambda}\ \mathrm{M}_1 \\ L' = L + e^{-2\lambda}\ \mathrm{M}_{1,r}\ \text{-}\ \lambda_{,r}\ \mathrm{e}^{-2\lambda}\ \mathrm{M}_1 \\ T' = T{+}1\ \frac{}{r}\ \mathrm{e}^{-2\lambda}\ \mathrm{M}_1 \\ H'_1 = \mathrm{H}_1 + \mathrm{M}_{0,r}\ \text{-}\ \nu_{,r}\ \mathrm{M}_0\ \mathrm{M}_{1,t} \\ h'_0 = \mathrm{h}_0 + \mathrm{M}_0 + \mathrm{M}_{2,t} \\ V' = V{+}\mathrm{M}\frac{2}{r^2} \\ h'_1 = \mathrm{h}_1\ \text{-}\ \mathrm{M}_{2,r} + \frac{2M_2}{r}\ \text{-}\ \mathrm{M}_1 \end{cases} \tag{7}$$

A possible choice of a polar gauge is to use the same asin the paper on stellar perturbations by Chandrasekhar and Ferrari, or in *The mathematical theory of black holes*, which imposes that M_0, M_1 and M_2 satisfy the equations :

$$\mathrm{H'}_1 = 0,\ \mathrm{h'}_0 = 0,\ \mathrm{h'}_1 = 0$$

If we let the same expression for the others coefficients, we finally find a polar perturbed metric tensor like :

$$\mathrm{h}_{lm}^{pol} = \begin{pmatrix} 2e^{2\nu}\ \mathrm{N}\ \mathrm{Y}_{lm} & 0 & 0 & 0 \\ 0 & -2r^2\ \sin^2\theta\ \mathrm{H}_{11} & 0 & -r^2\ \mathrm{V}\ \mathrm{X}_{lm} \\ 0 & 0 & -2e^{2\lambda}\ \mathrm{L}\ \mathrm{Y}_{lm} & 0 \\ 0 & -r^2\ \mathrm{V}\ \mathrm{X}_{lm} & 0 & -2r^2\ \mathrm{H}_{33} \end{pmatrix}$$

expressed in the basis $(e_t, e_\theta, e_r, e_\phi)$.

3.2.2 Axial part of the perturbed metric

Now, we consider the odd perturbations.

$$\begin{cases} h'_{tt} = \mathrm{h}_{tt} \\ h'_{t\phi} = \mathrm{h}_{t\phi} + \mathrm{M}_{3,t}\ \sin\theta\ \partial_\theta\ \mathrm{Y}_{lm} \\ h'_{tr} = \mathrm{h}_{tr} \\ h'_{t\theta} = \mathrm{h}_{t\theta}\ \text{-}\ \mathrm{M}_{3,t}\ \frac{1}{sin\theta}\ \partial_\phi\ \mathrm{Y}_{lm} \\ h'_{\phi\phi} = \mathrm{h}_{\phi\phi} + 2\ \mathrm{M}_3\ \sin\theta\ \partial_\theta\ \partial_\phi\ \mathrm{Y}_{lm}\ \text{-}\ 2\ \mathrm{M}_3\ \cos\theta\ \partial_\phi\ \mathrm{Y}_{lm} \\ h'_{r\phi} = \mathrm{h}_{r\phi} + \mathrm{M}_{3,r}\ \sin\theta\ \partial_\theta\ \mathrm{Y}_{lm}\ \text{-}\ \frac{2}{r}\ \mathrm{M}_3\ \sin\theta\ \partial_\theta\ \mathrm{Y}_{lm} \\ h'_{\phi\theta} = \mathrm{h}_{\phi\theta} + \mathrm{M}_3\ \sin\theta\ \partial_\theta^2\ \mathrm{Y}_{lm}\ \text{-}\ \mathrm{M}_3\ \cos\theta\ \partial_\theta\ \mathrm{Y}_{lm}\ \text{-}\ \mathrm{M}_3\ \frac{1}{sin\theta}\ \partial_\phi^2\ \mathrm{Y}_{lm} \\ h'_{rr} = \mathrm{h}_{rr} \end{cases} \tag{8}$$

With the same procedure used for the polar part, we can express the transformation of the three axial functions h_0, h_1 and h_2. Thus, we have :

$$\begin{cases} h'_0 = \mathrm{h}_0 + \mathrm{M}_{3,t} \\ h'_1 = \mathrm{h}_1 + \mathrm{M}_{3,r}\ \text{-}\ \frac{2}{r}\ \mathrm{M}_3 \\ h'_2 = \mathrm{h}_2\ \text{-}\ 2\ \mathrm{M}_3 \end{cases} \tag{9}$$

We can choose an arbitrary function to eliminate one of the three axial functions. So, arbitrarily, our axial gauge is $\mathrm{h'}_2 = 0$. And we obtain the following axial perturbed metric tensor :

$$\mathrm{h}_{lm}^{ax} = \begin{pmatrix} 0 & h_0^{ax}\ \sin\theta\partial_\theta Y_{lm} & 0 & -h_0^{ax}\ \frac{1}{sin\theta}\partial_\phi Y_{lm} \\ h_0^{ax}\ \sin\theta\partial_\theta Y_{lm} & 0 & h_1^{ax}\ \sin\theta\partial_\theta Y_{lm} & 0 \\ 0 & h_1^{ax}\ \sin\theta\partial_\theta Y_{lm} & 0 & -h_1^{ax}\ \frac{1}{sin\theta}\partial_\phi Y_{lm} \\ -h_0^{ax}\ \frac{1}{sin\theta}\partial_\phi Y_{lm} & 0 & -h_1^{ax}\ \frac{1}{sin\theta}\partial_\phi Y_{lm} & 0 \end{pmatrix}$$

3.3 Einstein's equations

Our initial perturbation $\mathrm{h}_{\mu\nu}$ can be considered like an axial perturbation $\mathrm{h}_{\mu\nu}^{ax}$ and a polar perturbation $\mathrm{h}_{\mu\nu}^{pol}$.
Our metric can thus be written : $\mathrm{g}_{\mu\nu} = \mathrm{g}_{\mu\nu}^0 + \mathrm{h}_{\mu\nu}^{ax} + \mathrm{h}_{\mu\nu}^{pol}$, with $\mathrm{g}_{\mu\nu}^0$ the Schwarzschild's metric.
In vacuum, at the first order in h, we have two systems of equations to solve :

$$\delta\mathrm{G}_{\mu\nu}\ (\mathrm{h}^{ax}) = 0,\ \delta\mathrm{G}_{\mu\nu}\ (\mathrm{h}^{pol}) = 0,$$

We recall
$\delta G_{\mu\nu}(h) = -\frac{1}{2}[h^{;\alpha}_{\mu\nu;\alpha} - (h^{;\alpha}_{\mu\alpha})_{;\nu} - (h^{;\alpha}_{\nu\alpha})_{;\mu} + 2\ R_{\alpha\mu\beta\nu}(g^0)h^{\alpha\beta} + h^{\alpha}_{\alpha;\mu\nu} - R^{\alpha}_{\nu}(g^0)h_{\mu\alpha} - R^{\alpha}_{\mu}(g^0)h_{\nu\alpha} + g^0_{\mu\nu}((h^{;\alpha}_{\mu\alpha})^{;\mu} - (h^{\alpha}_{\alpha;\beta})^{;\beta} - R^{\alpha\beta}(g^0)h_{\alpha\beta} + R(g^0)h_{\mu\nu}]$
If we make the Fourier transformation of the functions depending on time, the matrices h^{pol} and h^{ax} are exactly the same with coefficients depending on ω and r, and a coefficient $e^{i\omega t}$.
The general expression of the equations is :

$$D^{ax}_{\omega,r}[h^{ax}_0(\omega, r),\ h^{ax}_1(\omega, r)]\ O^{ax}_{\theta,\phi}\ Y_{lm}\ (\theta,\phi) = 0$$
$$D^{pol}_{\omega,r}[N^{pol}(\omega, r),\ V^{pol}(\omega, r),\ T^{pol}(\omega, r),\ L^{pol}(\omega, r)]\ O^{pol}_{\theta,\phi}\ Y_{lm}\ (\theta,\phi) = 0$$

With $D_{\omega,r}$ a differential operator in r and ω, and $O_{\theta,\phi}$ a differential operator in θ, ϕ acting on Y_{lm}.

The angular dependence is still present in our development. We need to eliminate it. To do this, we project these equations on the axial harmonics for the axial tensors, and on the polar harmonics for the polar tensors.

Finally, we obtain two sets of equations, one for the polar, the other for the axial perturbations, depending only on the frequency ω and on the radius r.

3.3.1 Regge-Wheeler equations for the axial perturbations

From the equation :

$$D^{ax}_{\omega,r}[h^{ax}_0(\omega, r),\ h^{ax}_1(\omega, r)]\ O^{ax}_{\theta,\phi}\ Y_{lm}\ (\theta,\phi) = 0$$

We obtain the following equations :
$-i\ \omega\ e^{-2\nu}\ h_0(\omega, r) - e^{-2\lambda}\ [h_{1,r}(\omega, r) + (\nu - \lambda)_{,r}\ h_1(\omega, r)] = 0$
$e^{-2\nu}\ [\ \omega^2\ h_{1,r}(\omega, r) - i\ \omega\ (h_{0,r}(\omega, r) - \frac{2}{r}\ h_0(\omega, r)\ )] - \frac{2n}{r^2}\ h_1(\omega, r) - 2^{-2\lambda}\ h_1(\omega, r)\ [\nu_{,rr} + (\frac{1}{r} + \nu_{,r})(\nu - \lambda)_{,r}] = 0$
Where $e^{2\nu} = (1 - \frac{r_s}{r})$ and $e^{2\lambda} = (1 - \frac{r_s}{r})^{-1}$.
We can simplify the second equation with $G_{\theta\theta} = 0$, and we have :
$\nu_{,rr} + (\frac{1}{r} + \nu_{,r})(\nu - \lambda)_{,r} = 0$
The axial equations become :

$$\left\{ \begin{array}{l} -i\omega\ e^{-2\nu}\ h_0(\omega, r) - e^{-2\lambda}\ [h_{1,r}(\omega, r) + (\nu - \lambda)_{,r}\ h_1(\omega, r)] = 0 \\ e^{-2\nu}\ [\ \omega^2\ h_{1,r}(\omega, r) - i\ \omega\ (h_{0,r}(\omega, r) - \frac{2}{r}\ h_0(\omega, r)\ )] - \frac{2n}{r^2}\ h_1(\omega, r) = 0 \end{array} \right. \quad (10)$$

Now, we differentiate the first equation and substitute in the second equation. We have :

$h_0 = [-2\nu_{,r}e^{4\nu}\ h_1 - e^{4\nu}\ h_{1,r}]\ \frac{1}{\omega}$,
$h_{0,r} = \frac{1}{\omega}\ [(-2\ \nu_{,rr}e^{4\nu} - 8\nu^2_{,r}e^{4\nu})\ h_1 - 6\ \nu_{,r}\ e^{4\nu}\ h_{1,r} - e^{4\nu}\ h_{1,rr}]$.

Hence,
$h_{1,rr}[-e^{2\nu}] + h_{1,r}[\frac{2}{r}e^{2\nu} - 6\nu_{,r}e^{2\nu}] + h_1[-\ \omega^2 e^{2\nu} - 2\nu_{,rr}e^{2\nu} - 8\nu^2_{,r}e^{2\nu} + \frac{4\nu_{,r}}{r}e^{2\nu} + \frac{l(l+1)}{r^2}] = 0.$
We set the function :

$$Z^{ax}\ (\omega, r) = \frac{1}{r}\ (1 - \frac{r_s}{r})\ h_1\ (\omega, r).$$

And a new radial coordinate, called the tortoise radius,

$$r_* = r + r_s \log(\frac{r}{r_s} - 1),$$

linked to the mass of the object as the Schwarzschild radius.
If we express the derivative with respect to r_* in function of the derivative with respect to r, we obtain : $\frac{d}{dr_*} = e^{2\nu}\ \frac{d}{dr}$.
We have :

$\frac{d^2 Z^{ax}}{dr_*^2} = e^{6\nu}\ [h_{1,rr}[\frac{1}{r}] + h_{1,r}[\frac{6\nu_{,r}}{r} - \frac{2}{r^2}] + h_1[\frac{8\nu^2_{,r}}{r} - \frac{6\nu_{,r}}{r^2} + \frac{2\nu_{,rr}}{r} + \frac{2}{r^3}]$
If we multiply the second order differential equation of h_1 by $\frac{-e^{4\nu}}{r}$, we obtain a second order differential equation for Z^{ax}, and :

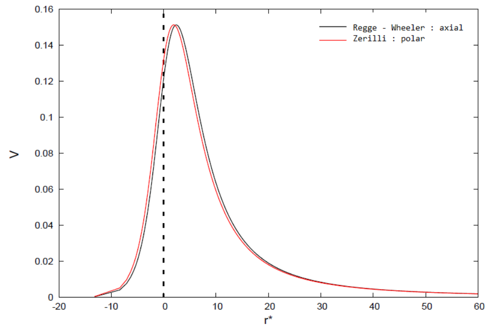

Figure 2: The two potentials, axial (black) and polar (red).

$\frac{d^2 Z^{ax}}{dr_*^2} + [\frac{\omega^2}{r} e^{-4\nu} - \frac{2+(l(l+1)-2)e^{-2\nu}}{r^3} + \frac{2\nu_{,r}}{r^2}]$ h_1 $e^{6\nu} = 0$.
Since we know that $e^{2\nu} = 1 - \frac{r_s}{r}$, we have $\nu_{,r} = - \frac{1}{2} \frac{r_s}{r^2} e^{-2\nu}$, it follows that :

$\frac{d^2 Z^{ax}}{dr_*^2} + [\frac{\omega^2}{r} e^{2\nu} + \frac{3r_s}{r^4} e^{4\nu} - \frac{l(l+1)}{r^3} e^{4\nu}]h_1 = 0$.
This is the **Regge-Wheeler equation** :

$$\frac{d^2 Z^{ax}(\omega,r)}{dr_*^2} + [\omega^2 - V^{ax}(r)] Z^{ax}(\omega, r) = 0$$
$$V^{ax}(r) = (1 - \frac{r_s}{r})(\frac{l(l+1)}{r^2} - \frac{3r_s}{r^3})$$

3.3.2 Zerilli equations for the polar perturbations

From the equation

$$D^{pol}_{\omega,r} [N(_\omega, r), T(_\omega, r), V(_\omega, r), L(_\omega, r)] O^{pol}_{\theta,\phi} Y_{lm} (\theta, \phi) = 0.$$

We can still separate the radial part and the angular part. We obtain the following set of equations.

$$\begin{cases} (T - V + N)_{,r} - (\frac{1}{r} - \nu_{,r}) N - (\frac{1}{r} + \nu_{,r}) L = 0 \\ V_{,r,r} + (\frac{2}{r} + \nu_{,r} - \lambda_{,r})V_{,r} + \frac{e^{2\lambda}}{r^2} (N + L) + \omega^2 e^{2\lambda - 2\nu} V = 0 \\ T - V + L = 0 \\ (\partial_r + \frac{1}{r} - \nu_{,r})(2T - l(l+1)V) - \frac{2}{r} L = 0 \\ e\frac{-2\lambda}{2}[\frac{2}{r}N_{,r} + (\frac{1}{r} + \nu_{,r})(2T - l(l+1)V)_{,r} - \frac{2}{r}(\frac{1}{r} + \\ 2\nu_{,r})L] + \frac{1}{2}[-\frac{1}{r^2}(2(l(l+1) - 2)T + l(l+1)N) + \omega^2 e^{-2\nu}(2T - l(l+1)V)] = 0 \end{cases} \quad (11)$$

We introduce the function :

$$Z^{pol}(\omega, r) = \frac{r}{\frac{l(l+1)}{2} r + 3M} [3MV - r L].$$

After similar appropriate manipulations, the polar equations can be reduced to the wave equation, the **Zerilli equation** :

$$\frac{d^2 Z^{pol}(\omega,r)}{dr_*^2} + [\omega^2 - V^{pol}(r)] Z^{pol}(\omega, r) = 0$$
$$V^{pol}(r) = \frac{2(r-2M)[n^2(n+1)r^3 + 3Mn^2r^2 + 9M^2nr + 9M^3]}{r^4(nr+3M)^2}$$
$$2n = (l-1)(l+2)$$
$$r_* = r + r_s \log(\frac{r}{r_s} - 1)$$

In conclusion, the axial and polar perturbations of a Schwarzschild black hole are described by a wave equation with different potential barriers. These two potentials are real and depend only on the radial position and on the black hole mass.

3.4 Quasi-normal modes

A perturbation can break the symmetry of the Schwarzschild space-time, and can also induce the emission of gravitational waves. The solutions of the Zerilli and the Regge-Wheeler equations in vacuum contain many informations about these gravitational waves. We can assume initially the emission depends totally of the perturbation, but after a very long time, is totally dependent of the characteristics of the black hole. We define the quasi-normal modes of a black hole as the complex solutions of the perturbation equations (Regge-Wheeler, Zerilli), verifying the asymptotic conditions :

$$\begin{cases} \mathrm{Z}^{pol,ax} \to Ae^{-i\omega r_*},\ r_* \to +\infty \Leftrightarrow \text{outgoing wave} \\ \mathrm{Z}^{pol,ax} \to Be^{i\omega r_*},\ r_* \to -\infty \Leftrightarrow \text{ingoing wave (nothing can go out of the horizon)} \end{cases} \tag{12}$$

The values of frequency for which the solution satisfy the imposed boundary conditions are the discrete set of the eigenvalues of the quasi-normal modes. Each frequency is complex, and can be written as : $\omega = \omega_R + \mathrm{i}\omega_I$. ω_R is the frequency of the mode oscillation, and we have to discuss ω_I. This term is the inverse of the damping time. Indeed the waves take energy from the black hole, and therefore the oscillations are damped. The case of $\omega_{I]}$ negative would mean that the amplitude of the mode explodes ; if that would happen, the black hole would be unstable. However, it has been proved that Schwarzschild's black holes are stable against the linear perturbations we consider. So we exclude this case.

Moreover, we can notice that the polar solution and the axial solution are isospectral, which means that the polar and the axial equations admit the same set of eigenvalues. This property is a particular characteristic of black holes. The amplitudes are linked by :

$$A^{pol}(\omega) = A^{ax}(\omega)\frac{n(n+1)-3iM\omega}{n(n+1)+3iM\omega}$$

So in vacuum, we can just solve one equation to know the other solution. We choose to solve the axial equation.

We set $\mathrm{Z}^{ax} = \mathrm{e}^{i\int^{r_*}\phi\,\mathrm{d}r_*}$, and the equation to solve becomes :

$$i\phi_{,r_*} + \omega^2 - \phi^2 - V^{ax} = 0$$

With the conditions : $\phi \to -\omega, r_* \to \infty$ and $\phi \to \omega, r_* \to -\infty$, which corresponds to the boundary conditions.
The solutions of a such problem exist only in the case where ω belongs to a discrete set of frequency. The works of Leaver tell us that the frequency of the quasi-normal modes depends only of the mass of the black hole, and ω_R is proportional to M whereas ω_I is proportional to $\frac{1}{M}$. If the black hole loses its symmetry, or if has a charge, the frequency would have others dependences.
In the next figure, we have computed some adimensioned quasi-normal modes $M\sigma = M\sigma_R + i\sigma_I$ of a Schwarzschild black hole. These results have driven Leaver to explain the preceeding paragraph.

n	l	$M\sigma_R + iM\sigma_I$	l	$M\sigma_R + iM\sigma_I$	l	$M\sigma_R + iM\sigma_I$
0	2	$0.37367 + i0.08896$	3	$0.59944 + i0.09270$	4	$0.80918 + i0.09416$
1	2	$0.34671 + i0.27391$	3	$0.58264 + i0.28130$	4	$0.79663 + i0.28443$
2	2	$0.30105 + i0.47828$	3	$0.55168 + i0.47909$	4	$0.77271 + i0.47991$
3	2	$0.25150 + i0.70514$	3	$0.51196 + i0.69034$	4	$0.73984 + i0.68392$

For example, if we take the first eigenvalue, $\sigma_R = \frac{0.37367}{M} = \frac{\omega_R}{c}$; the last equality yields from the definition of σ_R, since the dimension of the mass is a length (in fact, the "mass" is the Schwarzschild radius), and the dimension of σ_R is the inverse of a length.
We consider a black hole of n solar masses, M = n M_o, we obtain a frequency :

$$\nu_R = \frac{0.37C}{2\pi n M_o},$$

computing it, and knowing the Schwarzschild radius, or "solar mass", is $M_o = 1.5$ km, we have :

$$\nu_R = \frac{12}{n}\ \text{kHz}.$$

This is the frequency of the first quasi-normal mode. More the mass of the black hole is high, less the frequency is high. To detect the waves, we can use detector in several range of frequency :
-LIGO/VIRGO : 1.2 kHz
-LISA : 12 mHz

4 Perturbations induced by a particle falling into a black hole

4.1 Context

We consider a massive particle coming from radial infinity and falling in the gravitation field described by the Schwarzschild metric, along a geodesic. We set m_0 the mass of the particle. This one perturbs the black hole metric, and its stress-energy tensor is the source of Einstein's perturbed equations :

$$\delta G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

In the following development, we will set c = G = 1, like in the previous part.
The particle emits gravitational radiations as it falls until it is absorbed through the Schwarzschild's surface of radius 2M.
We need to set the boundary conditions, firstly, we will study the problem of the origin. In a Euclidean topology, we would require a regularity at the origin, but in the case of Schwarszchild, the origin is the curvature singularity, so we need to set a "boundary" condition on the events horizon. This surface is spacelike, so can be only crossed in one direction, in the sense of radius decreasing. So, we will assume that there are only ingoing waves on this surface. Nothing can come out of the black hole horizon.
The other boundary condition is at the infinity. We will look for outgoing waves solutions, such that the Euclidean topology would require. Indeed, at the infinity, this topology is available because the metric is reduce to Minkowski's metric.

4.2 Expression of the energy-momentum tensor

The source term is expressed by the energy-momentum tensor $T_{\mu\nu}$, given by an integral over a world line of the particle. The integrand contains a four-dimensional invariant δ function which is the guarantee of a divergence-free if the world line is a geodesic in the background geometry.

$$T^{\mu\nu} = m_0 \int \delta^{(4)}(x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \, d\tau$$

where $z(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau))$ is the quadriposition of the particle at the proper time τ, and $\frac{dz^\mu}{d\tau}$ are the components of the quadrivelocity of this particle along the geodesic. And we are still in our convention G = c =1.
We recall the normalization relation $\int_{spacetime} \delta^{(4)}(x)\sqrt{-g}\, d^4x = 1$.
With g, the determinant of Schwarzschild's metric.
In the following part, we will expand this tensor in tensors spherical harmonics.

4.3 Expression on the spherical harmonics basis

We expand $T^{\mu\nu}$ on the same basis used to expand $\delta G_{\mu\nu} = 0$.
We rewrite the basis of spherical harmonics, which depends only of the angles θ, ϕ :

$$\begin{cases} a_{lm} = [\boldsymbol{e_r}\ \boldsymbol{e_r}\ Y_{lm}] \\ b_{lm} = \sqrt{2}\ n(l)\ r\ [\boldsymbol{e_r}\ \boldsymbol{\nabla}\ Y_{lm}] \\ c_{lm} = \sqrt{2}\ n(l)\ r\ [\boldsymbol{e_r}\ L\ Y_{lm}] \\ d_{lm} = \sqrt{2}\ m(l)\ r\ ([L\ \boldsymbol{\nabla}\ Y_{lm}] + \frac{1}{r}\ [\boldsymbol{e_r}\ \boldsymbol{\nabla}\ Y_{lm}] \\ f_{lm} = \frac{1}{\sqrt{2}}\ m(l)\ (\boldsymbol{e_{lm}} + \boldsymbol{h_{lm}}) \\ g_{lm} = -\frac{1}{\sqrt{2}}\ n^2(l)\ (\boldsymbol{e_{lm}} - \boldsymbol{h_{lm}}) \\ a^{(0)}_{lm} = [\boldsymbol{e_t}\ \boldsymbol{e_t}\ Y_{lm}] \\ a^{(1)}_{lm} = \sqrt{2}\ [\boldsymbol{e_t}\ \boldsymbol{e_r}\ Y_{lm}] \\ b^{(0)}_{lm} = \sqrt{2}\ n(l)\ r\ [\boldsymbol{e_t}\ \boldsymbol{\nabla}\ Y_{lm}] \\ c^{(0)}_{lm} = \sqrt{2}\ n(l)\ [\boldsymbol{e_t}\ L\ Y_{lm}] \end{cases} \tag{13}$$

On this basis, after some calculations, we can express the coefficients of the energy-momentum tensor expansion, analogous to those given in equations (2) and (3) for the metric perturbations.

$$
\begin{cases}
A_{lm} = m_0\gamma \frac{dR}{dt}^2 (r-2M)^{-2}\delta(r-R(t))Y^*_{lm}(\Theta,\Phi) \\
B_{lm} = [\frac{1}{2}l(l+1)]^{-\frac{1}{2}} m_0\gamma(r-2M)^{-1}\frac{dR}{dt}\delta(r-R(t))\frac{dY^*_{lm}(\Theta,\Phi)}{dt} \\
Q_{lm} = [\frac{1}{2}l(l+1)]^{-\frac{1}{2}} im_0\gamma(r-2M)^{-1}\delta(r-R(t))\frac{dR}{dt}[\frac{1}{sin\Theta}\partial_\Phi Y^*_{lm}(\Theta,\Phi)\frac{d\Theta}{dt} - sin\Theta\partial_\Theta Y^*_{lm}(\Theta,\Phi)\frac{d\Phi}{dt}] \\
D_{lm} = -[\frac{1}{2}l(l+1)(l-1)(l+2)]^{-\frac{1}{2}} im_0\gamma\delta(r-R(t))[\frac{1}{2}[\frac{d\Theta}{dt}^2 - sin^2\Theta\frac{d\Phi}{dt}^2]\frac{1}{sin\Theta}X^*_{lm}(\Theta,\Phi) - sin\Theta\frac{d\Phi}{dt}\frac{d\Theta}{dt}W^*_{lm}(\Theta,\Phi)] \\
F_{lm} = [\frac{1}{2}l(l+1)(l-1)(l+2)]^{-\frac{1}{2}} m_0\gamma\delta(r-R(t))[\frac{d\Theta}{dt}\frac{d\Phi}{dt}X^*_{lm}(\Theta,\Phi) + \frac{1}{2}[\frac{d\Theta}{dt}^2 - sin^2\Theta\frac{d\Phi}{dt}^2]W^*_{lm}(\Theta,\Phi)] \\
G_{lm} = \frac{m_0\gamma}{sqrt2}\delta(r-R(t))[\frac{d\Theta}{dt}^2 + sin^2\frac{d\Phi}{dt}^2]Y^*_{lm}(\Theta,\Phi) \\
A^{(0)}_{lm} = m_0\gamma(1-\frac{2M}{r})^2 r^{-2}\delta(r-R(t))Y^*_{lm}(\Theta,\Phi) \\
A^{(1)}_{lm} = sqrt2im_0\gamma\frac{dR}{dt}r^{-2}\delta(r-R(t))Y^*_{lm}(\Theta,\Phi) \\
B^{(0)}_{lm} = [\frac{1}{2}l(l+1)]^{-\frac{1}{2}} im_0\gamma(1-\frac{2M}{r})r^{-1}\delta(r_R(t))\frac{dY^*_{lm}(\Theta,\Phi)}{dt} \\
Q^{(0)}_{lm} = [\frac{1}{2}l(l+1)]^{-\frac{1}{2}} m_0\gamma(1-\frac{2M}{r})r^{-1}\delta(r-R(t))[\frac{1}{sin\Theta}\partial_\Phi Y^*_{lm}(\Theta,\Phi)\frac{d\Theta}{dt} - sin\Theta\partial_\Theta Y^*_{lm}(\Theta,\Phi)\frac{d\Phi}{dt}]
\end{cases}
\tag{14}
$$

Where $\gamma = \frac{dT(\tau)}{d\tau}$, and $X_{lm} = 2\partial_\Phi(\partial_\Theta - cot\Theta)Y_{lm}$, $W_{lm} = (\partial^2_\Theta - cot\Theta\partial_\Theta - \frac{1}{sin^2\Theta}\partial^2_\Phi)Y_{lm}$
As for the expansion of $h_{\mu\nu}$, also for $T_{\mu\nu}$ we have a polar part and an axial part. We have used the same names of the coordinates for polar part and axial part, and thus : the polar part is expressed by $A_{lm}, A^{(0)}_{lm}, A^{(1)}_{lm}, B_{lm}, B^{(0)}_{lm}, F_{lm}$ and G_{lm}, and the axial part is expressed by Q_{lm}, $Q^{(0)}_{lm}$ and D_{lm}.
Now we make the Fourier transform of these quantities by multiplying by $\frac{1}{\sqrt{2\pi}}e^{i\omega t}$.

4.4 Einstein's equations

We study a perturbation of the metric $g^0_{\mu\nu}$, $h_{\mu\nu}$. Since the infalliing particle acts as a perturbation our metric becomes $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$. Seen previously, $G_{\mu\nu}(g^0_{ab}) = 0$, so in the natural units at the first order, $G_{\mu\nu}(g_{ab}) = G_{\mu\nu}(g^0_{ab}) + \delta G_{\mu\nu}(h_{ab})$, and since in a general case, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we obtain :

$$\delta G_{\mu\nu}(h_{ab}) = 8\pi T_{\mu\nu}.$$

The last equation can be written with the Ricci tensor $R_{\mu\nu}$, and the determinant T of the stress-energy tensor (see annex 6.4).

$$\delta R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g^0_{\mu\nu}T).$$

Now, we can use the preceeding parts about the calculation of the perturbed Einstein tensor.
We will study separately the polar case and the axial case.

4.4.1 Polar case

The line-element with an axial perturbation can be written, in the Chandrasekhar-Ferrari gauge ($h_2 = 0, H_1 = 0, h_1 = 0, h_2 = 0$) as :

$$ds^2 = e^{2\nu}[1+2NY_{lm}]dt^2 - e^{-2\nu}[1+2LY_{lm}]dr^2 - r^2[1+2H_{11}Y_{lm}]d\theta^2 - r^2sin^2\theta[1+2H_{33}Y_{lm}]d\phi^2 - 2r^2VX_{lm}Y_{lm}d\theta d\phi$$

Where $e^{2\nu} = 1 - \frac{r_s}{r} = 1 - \frac{2M}{r}$, M the black hole mass. X_{lm}, H_{11} and H_{33} are defined in the subsection 3.1.
We assume that the temporal dependence of the perturbation is $e^{-i\omega t}$. This comes from the Fourier transform.

The components of $\delta R_{\mu\nu}$ and $\delta G_{\mu\nu}$ relevant for the polar perturbations are, after computing :

$$
\begin{cases}
\delta R_{02} = i\omega Y_{lm}(\frac{d}{dr} + \frac{1}{r} - \nu_{,r})(2T - l(l+1)V - \frac{2L}{r}) \\
\delta R_{03} = -i\omega Y_{lm,\theta}(V - T - L) \\
\delta R_{23} = -Y_{lm,\theta}[(T - V + N)_{,r} - (\frac{1}{r} - \nu_{,r})N - (\frac{1}{r} + \nu_{,r})L] \\
\delta R_{11} = (Y_{lm} + sin^2\theta Y_{lm} - \frac{sin2\theta}{2}Y_{lm,theta})[r^2e^{2\nu}[\nu_{,rr} + 2(\frac{1}{r} + \nu_{,r})V_{,r} + \frac{e^{-2\nu}}{r^2}(N+L) + \omega^2e^{-4\nu}V] \\
\delta G_{22} = Y_{lm}[\frac{2}{r}N_{,r} + (\frac{1}{r} + \nu_{,r})(2T - l(l+1)V)_{,r}l(l+1)\frac{e^{-2\nu}}{r^2}N - \frac{2ne^{-2\nu}}{r^2}T \\
+\omega^2e^{-4\nu}(2T - l(l+1)V) - \frac{2}{r}(\frac{1}{r} + 2\nu_{,r})L]
\end{cases}
\tag{15}
$$

And the corresponding components of $T_{\mu\nu}$ are :

$$\begin{cases} T_{02} = \frac{Y_{lm}}{\sqrt{2}} A^{(1)}_{lm} \\ T_{22} = Y_{lm} A_{lm} \\ T_{03} = \frac{n(l)}{\sqrt{2}} B^{(0)}_{lm} Y_{lm,\theta} \\ T_{23} = \frac{n(l)r}{\sqrt{2}} B_{lm} Y_{lm,\theta} \\ T_{11} = \frac{r^2}{\sqrt{2}} (G_{lm} sin^2\theta Y_{lm} - m(l) W_{lm} F_{lm}) \end{cases} \tag{16}$$

With n(l), m(l) and W_{lm} defined previously.

Writing Einstein's equations, we obtain the following set of equation :

$$\begin{cases} (\partial_r + \frac{1}{r} - \nu_{,r})(2T - l(l+1)V - \frac{2L}{r}) = \frac{8\pi A^{(1)}_{lm}}{i\omega\sqrt{2}} \equiv S_{A^1} \\ T + L - V = 8\frac{\pi r B^{(0)}_{lm}}{i\omega\sqrt{2l(l+1)}} \equiv S_{B^0} \\ (T - V + N)_{,r} - (\frac{1}{r} - \nu_{,r})N - (\frac{1}{r} + \nu_{,r})L] = -\frac{8\pi r B_{lm}}{\sqrt{2l(l+1)}} \equiv S_B \\ [2N\frac{,r}{r + (\frac{1}{r} + \nu_{,r})(2T - l(l+1)V)_{,r} l(l+1)\frac{e^{-2\nu}}{r^2} N - \frac{2ne^{-2\nu}}{r^2} T + \omega^2 e^{-4\nu}(2T - l(l+1)V) - \frac{2}{r}(\frac{1}{r} + 2\nu_{,r})L]} = 8\pi A_{lm} \equiv S_A \\ \nu_{,rr} + 2(\frac{1}{r} + \nu_{,r})V_{,r} + \frac{e^{-2\nu}}{r^2}(N + L) + \omega^2 e^{-4\nu} V = \frac{16\pi r F_{lm}}{(r - r_s)\sqrt{2l(l-1)(l+1)(l+2)}} \equiv S_F \end{cases} \tag{17}$$

We notice that the second, the fourth and the last equations are exactly the same found in the case of the vacuum in the subsection 3.3.2. Here, we have a source term, represented by S_{B^0}, S_A and S_F.

Zerilli equation with source term We can express these equations with the same transformations than for the development of the Zerilli equation in the vacuum.

$$\frac{d^2 Z^{pol}(\omega,r)}{dr_*^2} + [\omega^2 - V^{pol}(r)]\, Z^{pol}(\omega, r) = S^{pol}_{lm}$$

Where S^{pol} is null in the case of the vacuum, and its expression will be defined in the next step.
With the transformations driving from the set of five equations to the Zerilli equation (we don't write them, because they are too heavy), we obtain :

$$S^{pol}_{lm} = \frac{(r-2M)^2}{2(nr+3M)} S_A - \frac{(r-2M)^2}{r(nr+3M)} S_B + \frac{M(r-2M)(nr+3r-3M)}{r(nr+3M)^2} S_{A^1} + \frac{(r-2M)^2}{2(nr+3M)} S_{A^1_{,r}} - \frac{r-2M}{r}[\frac{(n+1)(nr-3M)}{(nr+3M)^2} - \frac{\omega^2 r^3}{(r-2M)(nr+3M)}] S_{B^0} - \frac{(r-2M)(nr^2+9Mr-12M^2)}{r(nr+3M)^2} S_{B^0_{,r}} - \frac{(r-2M)^2}{nr+3M} S_{B^0_{,rr}} + \frac{(r-2M)^2}{r} S_F$$

4.4.2 Axial case

For the axial case, we will follow exactly the same procedure as for the polar case in the Chandrasekhar-Ferrari gauge.
The line-element is expressed by :

$$ds^2 = 2h_0 sin\theta Y_{lm,\theta} dt d\phi + 2h_1 sin\theta Y_{lm,\theta} dr d\phi - 2h_0 \frac{Y_{lm,\phi}}{sin\theta} dt d\theta - 2h_1 \frac{Y_{lm,\phi}}{sin\theta} dt d\theta$$

We write the dependence in $e^{im\phi}$ of Y_{lm} by the Legendre polynomials, so $Y_{lm,\phi} = imY_{lm}$.

We express the terms in $_{r\phi}$ and $_{\theta\phi}$ for Einstein's equations. And we obtain the following system of equations :

$$\begin{cases} d\frac{^2h_1}{dt^2} - \frac{d^2h_0}{dtdr} + \frac{2}{r}\frac{dh_0}{dt} + \frac{2ne^{2\nu}}{r^2} h_1 = 16\pi \frac{in(l)re^{2\nu}}{\sqrt{2}} Q_{lm} \\ e^{2\nu}\frac{dh_1}{dr} + \frac{2M}{r^2} h_1 - e^{-2\nu}\frac{dh_0}{dt} = 16\pi \frac{im(l)r^2}{\sqrt{2}} D_{lm} \end{cases} \tag{18}$$

Regge-Wheeler equation with source term If we apply the same transformations than for finding the Regge-Wheeler equation, we have the Regge-Wheeler equation with a source term.

$$\frac{d^2 Z^{ax}(\omega,r)}{dr_*^2} + [\omega^2 - V^{ax}(r)]\, Z^{ax}(\omega, r) = S^{ax}_{lm}$$

Where $S^{ax}_{lm} = \frac{16\pi i e^{2\nu}}{r\sqrt{2l(l-1)(l+1)(l+2)}}[r^2(e^{2\nu} D_{lm})_{,r} - \sqrt{(l-1)(l+2)} r e^{2\nu} Q_{lm}]$

4.4.3 Example : the case of a pointmass particle in radial infall

If we have a point particle of mass m_0 in a radial infall, coming from the infinity and arriving at r = r_s, its angular momentum is null. Moreover, in the Schwarzschild's space-time, the trajectories are plane, so, we can take $\theta = \frac{\pi}{2}$, the equatorial plane.
The geodesics are solution of the following set of equations coming from the equation of Euler-Lagrange :

$$\begin{cases} d\frac{\phi}{d\tau} = 0 \\ d\frac{\theta}{d\tau} = 0 \\ dt\frac{}{d\tau} = E(1 - \frac{2M}{r})^{-1} \\ dr\frac{}{d\tau} = -\sqrt{E^2 - 1 + \frac{2M}{r}} \end{cases} \tag{19}$$

With τ, the proper time.
Since the angular momentum is null and it is conserved on the geodesics, giving the coordinates on the spherical harmonics basis, we have only A_{lm} and $A_{lm}^{(1)}$ non null. All the others are proportionnals to the angular momentum. Thus, we have :

$A_{lm} = \frac{\sqrt{2}}{r(r-2M)}\Theta_{lm}(0)e^{f(r)}$,
$A_{lm}^{(1)} = \frac{1}{(r-2M)^2}\sqrt{g(r)}\Theta_{lm}(0)e^{f(r)}$.
With g(r) = $\frac{2M}{r}$ and f(r) = $-i[\omega t(r) + m\phi t(r)]$.

Along a geodesic, t(r) is given by :
t(r) = $-2M[\frac{2}{3}\frac{r}{2M}^{3/2} + 2\frac{r}{2M}^{1/2} + ln\frac{(r/2M)^{1/2}+1}{(r/2M)^{1/2}-1}]$.
Finally, we have for the source term only the polar part,

$S_{lm} = S_{lm}^{pol} = -(1 - \frac{2M}{r})\frac{4m_0}{nr+3M}\sqrt{l + \frac{1}{2}}[\sqrt{\frac{r}{2M}} + \frac{2in}{\omega(nr+3M)}]e^{-i\omega t(r)}$.
This means that a particle which falls radially in a Schwarzschild black hole does not excite the axial perturbations.

5 Conclusion

We have seen since the background is spherically symmetric, we have to expand every quantity on the tensors spherical harmonics basis for simplicity.
Moreover, on this basis, we can decompose any perturbation in two parts : one polar, the other axial, and study them separately.
Once done, we are able to define two equations describing the polar and the axial perturbations, respectively Zerilli and Regge-Wheeler. In these two equations, we have defined two potentials, very similar, which depend of the mass black hole. A such equation permit to compute the quasi-normal modes of the waves, which are totally dependent of the black hole characteristics.
These modes are antiproportional to the mass of their source, and we are able to compute a range of frequencies. This range permits to develop different kind of interferometers like LISA or LIGO, to detect waves coming from a specific black-hole. For example, to probe the black hole of our center galactic, we are expecting a frequency for the first mode of 3 mHz, according to the expression of ν_R.
We have also seen that a perturbation breaking the symmetry of the black hole is at the origin of the wave emission. A such perturbation can be a simple particle falling radially into the black hole, and the source term of the wave is totally dependent of the stress-energy tensor of the particle.
The waves detected come from a black hole of 30-40 solar masses, thus, a frequency for the first mode of 300-400 Hz, which is in the range of the detector used, LIGO, 30-7000 Hz.

6 Annex

6.1 Einstein's and Ricci's tensors

If we study the Riemann and the Ricci tensors, we can see the both are construted linearly in the second derivative of $g_{\mu\nu}$, like $G_{\mu\nu}$. So, we can look for a linear combination : $G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R$.
Making the covariant derivative, we obtain :
$G^{\mu\nu}_{;\mu} = C_1 R^{\mu\nu}_{;\mu} + C_2 g^{\mu\nu} R_{;\mu} = 0$, according the conservation law of the stress-energy tensor.
Indeed, $g^{\mu\nu}_{;\mu} = 0$ according to the equivalence principle. We have locally $g_{ab;\mu} = \eta_{ab;\mu} = \eta_{ab,\mu}$ - $\Gamma^{\nu}_{a\mu}\eta_{b\nu} - \Gamma^{\nu}_{b\mu}\eta_{a\nu} = 0$, because the metric η_{ab} is a constant and the Cristoffel symbol Γ is zero in a flat space.

The Bianchi identity is : $R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$.
Contracting it with $g^{\lambda\nu}$, we obtain : $R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0$, since $R_{abcd} = -R_{abdc}$.
Contracting again with $g^{\mu\kappa}$, we obtain : $R_{;\eta} - R^{\kappa}_{\eta;\kappa} - R^{\nu}_{\eta;\nu} = 0$.
The last equation can be written $(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$. And we obtain, $C_1 = -2C_2$.

To compute C_1, we use the weak field limit : $|T_{ij}| << |T_{00}|$, with i,j = 1,3, so $|G_{ij}| << |G_{00}|$, and $|C_1(R_{ij} - \frac{1}{2}g_{ij}R| << |G_{00}|$. Hence, $R_{ij} = \frac{1}{2}g_{ij}R$.
In this limit, at the first order, $g_{ij} = \eta\prime_{ij}$, and so $2R_{kk}$ = R = -R_{00} + $\frac{3}{2}R$.
We obtain $G_{00} = 2C_1 R_{00}$,
with $R_{00} = -\frac{1}{2}\eta^{ij}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j} = -\frac{1}{2}\nabla^2 g_{00}$, at the first order.
Hence, G_{00} = -$C_1\nabla^2 g_{00}$. Moreover, by the Laplace genralized equation -$\nabla^2 g_{00} = 8\pi T_{00} = G_{00}$, and thus $C_1 = 1$.
It yields, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

6.2 Expression of the perturbed Einstein tensor

From the expression of g^{ij} and g_{ij} developed in sections 2.1.2, we can compute the Ricci tensor :

$$2\,\mathrm{R}_{ij} = g^{0ab}g^0_{aj,bi} + g^{0ab}h_{aj,bi} - h^{ab}g^0_{aj,bi} - g^{0ab}g^0_{ai,bj} - g^{0ab}h_{ai,bj} + h^{ab}g^0_{ai,bj} + g^{0ab}g^0_{bi,aj} + g^{0ab}h_{bi,aj} - h^{ab}g^0_{bi,aj} - g^{0ab}g^0_{bj,ai} - g^{0ab}h_{bj,ai} - h^{ab}g^0_{bj,ai} + O(h^2)$$

And :

$$2\,\mathrm{g}_{ij}\mathrm{R} = g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}h_{\alpha\nu,\beta\mu} - g^0_{ij}g^{0\mu\nu}h^{\alpha\beta}g^0_{\alpha\nu,\beta\mu} - g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}h_{\alpha\nu,\beta\mu} + g^0_{ij}g^{0\mu\nu}h^{\alpha\beta}g^0_{\alpha\mu,\beta\nu} + g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}h_{\beta\mu,\alpha\nu} - g^0_{ij}g^{0\mu\nu}h^{\alpha\beta}g^0_{\beta\mu,\alpha\nu} - g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}h_{\beta\nu,\alpha\mu} - g^0_{ij}g^{0\mu\nu}h^{\alpha\beta}g^0_{\beta\mu,\alpha\nu} - g^0_{ij}h^{\mu\nu}g^{0\alpha\beta}g^0_{\alpha\nu,\beta\mu} + g^0_{ij}h^{\mu\nu}g^{0\alpha\beta}g^0_{\alpha\mu,\beta\nu} - g^0_{ij}h^{\mu\nu}g^{0\alpha\beta}g^0_{\beta\mu,\alpha\nu} + g^0_{ij}h^{\mu\nu}g^{0\alpha\beta}g^0_{\beta\nu,\alpha\mu} + h_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\alpha\mu,\beta\nu} - h_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\alpha\mu,\beta\nu} + h_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\beta\mu,\alpha\nu} - h_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\beta\nu,\alpha\mu} + g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\alpha\mu,\beta\nu} - g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\alpha\mu,\beta\nu} + g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\beta\mu,\alpha\nu} - g^0_{ij}g^{0\mu\nu}g^{0\alpha\beta}g^0_{\beta\nu,\alpha\mu} + \mathrm{O}(\mathrm{h}^2)$$

The terms containing only the Schwarszchild's metric and not the perturbation are the terms which correspond to the equation $\mathrm{G}_{\mu\nu}(g^0) = 0$, ans thus, we can delete them to obtain the complete equation δ $\mathrm{G}_{\mu\nu}$(h) = 0, which are linear in $\mathrm{h}_{\mu\nu}$ at the first order.
With these quantities, we can find an explicit expression for $\delta\mathrm{G}_{\mu\nu}$(h), forgetting after have computing the explicit expression using the expressions of the Ricci's tensor and the curvature,

$\delta\mathrm{G}_{\mu\nu}(\mathrm{h})$ = -$\frac{1}{2}$[$\mathrm{h}^{;\alpha}_{\mu\nu;\alpha}$ - $(\mathrm{h}^{;\alpha}_{\mu\alpha})_{;\nu}$ - $(\mathrm{h}^{;\alpha}_{\nu\alpha})_{;\mu}$ + 2 $\mathrm{R}_{\alpha\mu\beta\nu}(\mathrm{g}^0)\mathrm{h}^{\alpha\beta}$ + $\mathrm{h}^{\alpha}_{\alpha;\mu\nu}$ - $\mathrm{R}^{\alpha}_{\nu}(\mathrm{g}^0)\mathrm{h}_{\mu\alpha}$ - $\mathrm{R}^{\alpha}_{\mu}(\mathrm{g}^0)\mathrm{h}_{\nu\alpha}$ + $\mathrm{g}^0_{\mu\nu}((\mathrm{h}^{;\alpha}_{\mu\alpha})^{;\mu}$ - $(\mathrm{h}^{\alpha}_{\alpha;\beta})^{;\beta}$ - $\mathrm{R}^{\alpha\beta}(\mathrm{g}^0)\mathrm{h}_{\alpha\beta}$ + $\mathrm{R}(\mathrm{g}^0)\mathrm{h}_{\mu\nu}$].

6.3 Spherical harmonics

Given a scalar function f $\longrightarrow$ f(**r**, $\boldsymbol{\theta}$, $\boldsymbol{\phi}$), a rotation of an angle dϕ around the z-axis creates a new field, according to the Taylor expansion at the first order.

$$\mathrm{f}(\mathbf{r}, \boldsymbol{\phi} - \mathrm{d}\boldsymbol{\phi}, \boldsymbol{\theta}) = (1 + \mathrm{d}\phi\ \partial_{\phi})\ \mathrm{f}(\mathbf{r}, \boldsymbol{\phi}, \boldsymbol{\theta}).$$

We set L_z = - i ∂_ϕ , which is an operator describing infinitesimal rotations around z-axis. Similarly, we define L_x and L_y. These are the components of the angular momentum operator L = - i **r** x $\boldsymbol{\nabla}$ with $\hbar = 1$, and ∇, the covariant derivative.
We need to compute the commutator [L_i , L_j], for i, j $\in$ [x, y, z].
We have

$$\begin{cases} L_x = \text{- i } r_y \nabla_z + \text{i } r_z \nabla_y \\ L_y = \text{- i } r_x \nabla_z + \text{i } r_z \nabla_x \\ L_z = \text{- i } r_x \nabla_y + \text{i } r_y \nabla_x \end{cases} \tag{20}$$

Computing the different commutators,

$[L_x , L_y] = [- i\, r_y \nabla_z + i\, r_z \nabla_y , - i\, r_x \nabla_z + i\, r_z \nabla_x]$
$[L_x , L_y] = [- i\, r_y \nabla_z , - i\, r_z \nabla_x]$ - $[- i\, r_y \nabla_z , - i\, r_x \nabla_z]$ - $[- i\, r_z \nabla_y , - i\, r_z \nabla_y]$ + $[- i\, r_z \nabla_y , - i\, r_x \nabla_z]$
$[L_x , L_y] = - i\, r_y \nabla_x [- i \nabla_z , r_z] - 0 - 0 - i\, r_x \nabla_y [r_z , - i \nabla_z]$

At this point, we need to recall that [r_i , p_j] = i $\hbar$ δ_{ij}, and p_j = - i $\hbar$ ∇_j.
So,
[L_x , L_y] = i $\hbar$ L_z.

Similarly,
[L_y , L_z] = i $\hbar$ L_x and [L_x , L_z] = - i $\hbar$ L_y

We can summarize these three relations by :

$$[L_i , L_j] = i\, \epsilon_{ijk}\, L_k$$

with ϵ_{ijk}, the Levi-Civita pseudotensor which is equal to 1 if the permutation between i, j and k is even, -1 if it is odd and 0 if two indices are equal.

The operator $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with all the components of L. Thus, we can find a complete basis of eigenfunctions where L_i and L^2 are diagonalisable. In spherical coordinates, we have :
L_z = - i ∂_ϕ,
L_x = i (sin ϕ ∂_θ + cot θ cos ϕ ∂_ϕ),
L_y = i (- cot ϕ ∂_θ + cot θ sin ϕ ∂_ϕ).

Hence, we obtain

$$L^2 = - [\frac{1}{sin^2\theta}\, \partial_\phi^2 + \frac{1}{sin\theta}\, \partial_\theta(sin\theta\, \partial_\theta)].$$

If we solve the following system, we can find the eigenvalues.

$$\begin{cases} L_z = \text{m } Y_{lm} \\ L^2 = \text{l(l+1) } Y_{lm} \end{cases} \tag{21}$$

where Y_{lm} are the eigenfunctions of L_z and L^2 with respective eigenvalues m and l(l+1). From the expression of L^2, L_z and the preeceding system, we obtain :

$$Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta)\, e^{im\phi}$$

and we can assume that l is an arbitrary integer, and m $\in$ [-l , l] integer. The explicit expression of Θ is the following.

$$\Theta\ (\theta) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\ P_{lm}\ (\cos\theta).$$

In this expression, P_{lm} is the associated Legendre polynomial, solution of

$$[\partial_\theta^2 + \cot\theta\ \partial_\theta + l(l+1) - \frac{m^2}{sin^2\theta}]\ P_{lm}\ (\cos\theta) = 0$$

We have the following relation among the Legendre functions :

$$P_{l,-m}\ (\cos\theta) = (-1)^m\ \sqrt{\frac{(l-m)!}{(l+m)!}}\ P_{lm}\ (\cos\theta)$$

So, Y_{lm} and its complex conjugate are linked by

$$Y^*_{lm}\ (\theta, \phi) = Y_{lm}\ (\theta, -\phi) = (-1)^m\ Y_{l,-m}\ (\theta, \phi)$$

The orthonormality condition is

$$\int_0^{2\pi} d\phi \int_0^{\pi} Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) sin\theta\, d\theta = \delta_{mm'}\delta_{ll'}$$

and the completeness relation is

$$\Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}\ Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) = \frac{1}{sin\theta}\delta(\phi-\phi')\delta(\theta-\theta')$$

Thus, the basis of spherical harmonics Y_{lm} is a complete, orthonormal basis on the 2-sphere euclidean space. It is totally appropriated to describe our system.

6.4 Quadrivectors

Now, we can use the basis of spherical vector harmonics to construct the basis of quadrivectors. To do that, we set the following matricial equations :

$$\begin{cases} v^{(0)} = \boldsymbol{e_t}\ Y_{lm} \\ v^{(1)} = L\ Y_{lm} \\ v^{(2)} = \nabla\ Y_{lm} \\ v^{(3)} = \boldsymbol{e_r}\ Y_{lm} \end{cases} \tag{22}$$

Where e_r is the unit radial vector equal to (0, 0, 0, 1), e_t is the unit time vector (1, 0, 0, 0), L the angular momentum operator and ∇ the covariant derivative. We project Y_{lm} on the spherical coordinates basis (e_t, L, ∇, e_r) to find the different quadrivectors.
Hence,

$$\begin{cases} v^{(0)} = (Y_{lm}, 0, 0, 0) \\ v^{(1)} = (0, i\ r\ \sin\theta\ \partial_\theta\ Y_{lm}, 0, -\ i\ r\ \frac{1}{sin\theta}\partial_\phi\ Y_{lm}) \\ v^{(2)} = (0, -\ \partial_\phi\ Y_{lm}, 0, -\ \partial_\theta\ Y_{lm}) \\ v^{(3)} = (0, 0, -\ Y_{lm}, 0) \end{cases} \tag{23}$$

Given any quadrivector f(t, r, θ, ϕ), its expression in the harmonics basis is :

$$f(t, r, \theta, \phi) = \Sigma_{lm}\ (f^{(0)}_{lm}\ v^{(0)} + f^{(1)}_{lm}\ v^{(1)} + f^{(2)}_{lm}\ v^{(2)} + f^{(3)}_{lm}\ v^{(3)})$$

With $f^{(i)}_{lm} = \int v^{(0)*}_{lm}{}_{\mu} v^{\mu} sin\theta\, d\theta d\phi$
Thus, each function f_{lm} depends only of the time coordinate t, and the radial position r, not of the angles.

This system form a complete basis of spherical harmonics for the quadrivectors. We can extend every quadrivector on this basis.

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